Table 7.2 summarizes the methods used to integrate $\int \tan^m x \sec^n x\,dx$. Analogous techniques are used for $\int \cot^m x \csc^n x\,dx$.

Table 7.2

$\int \tan^m x \sec^n x\,dx$	Strategy
n even	Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
m odd	Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
m even and n odd	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

SECTION 7.2 EXERCISES

Review Questions

1. State the half-angle identities used to integrate $\sin^2 x$ and $\cos^2 x$.
2. State the three Pythagorean identities.
3. Describe the method used to integrate $\sin^3 x$.
4. Describe the method used to integrate $\sin^m x \cos^n x$ for m even and n odd.
5. What is a reduction formula?
6. How would you evaluate $\int \cos^2 x \sin^3 x\,dx$?
7. How would you evaluate $\int \tan^{10} x \sec^2 x\,dx$?
8. How would you evaluate $\int \sec^{12} x \tan x\,dx$?

Basic Skills

9–12. Integrals of sin x or cos x *Evaluate the following integrals.*

9. $\int \sin^2 x\,dx$
10. $\int \cos^4 2x\,dx$
11. $\int \sin^5 x\,dx$
12. $\int \cos^3 20x\,dx$

13–18. Integrals of sin x and cos x *Evaluate the following integrals.*

13. $\int \sin^2 x \cos^2 x\,dx$
14. $\int \sin^3 x \cos^5 x\,dx$
15. $\int \sin^5 x \cos^{-2} x\,dx$
16. $\int \sin^{-3/2} x \cos^3 x\,dx$
17. $\int \sin^2 x \cos^4 x\,dx$
18. $\int \sin^3 x \cos^{3/2} x\,dx$

19–24. Integrals of tan x or cot x *Evaluate the following integrals.*

19. $\int \tan^2 x\,dx$
20. $\int 6\sec^4 x\,dx$
21. $\int \tan^3 4x\,dx$
22. $\int \sec^5 \theta\,d\theta$
23. $\int 20\tan^6 x\,dx$
24. $\int \cot^5 3x\,dx$

25–32. Integrals of tan x and sec x *Evaluate the following integrals.*

25. $\int \sec^2 x \tan^{1/2} x\,dx$
26. $\int \sec^{-2} x \tan^3 x\,dx$
27. $\int \dfrac{\csc^4 x}{\cot^2 x}\,dx$
28. $\int \csc^{10} x \cot^3 x\,dx$
29. $\int_0^{\pi/4} \sec^4 \theta\,d\theta$
30. $\int \tan^5 \theta \sec^4 \theta\,d\theta$
31. $\int_{\pi/6}^{\pi/3} \cot^3 \theta\,d\theta$
32. $\int_0^{\pi/4} \tan^3 \theta \sec^2 \theta\,d\theta$

Further Explorations

33. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
 a. If m is a positive integer, then $\int_0^{\pi} \cos^{2m+1} x\,dx = 0$.
 b. If m is a positive integer, then $\int_0^{\pi} \sin^m x\,dx = 0$.

34–37. Integrals of cot x, sec x, and csc x

34. Use a change of variables to prove that $\int \cot x\,dx = \ln|\sin x| + C$.
35. Prove that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. (*Hint:* Multiply numerator and denominator of the integrand by $\sec x + \tan x$; then make a change of variables with $u = \sec x + \tan x$.)
36. Prove that $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$. (*Hint:* Use a method analogous to that used in Exercise 35.)
37. Use the results of Theorem 7.1 to find the indefinite integral of $\tan ax$ and $\sec ax$, where a is a nonzero real number.
38. **Comparing areas** The region R_1 is bounded by the graph of $y = \tan x$ and the x-axis on the interval $[0, \pi/3]$. The region R_2 is bounded by the graph of $y = \sec x$ and the x-axis on the interval $[0, \pi/6]$. Which region has the greater area?
39. **Region between curves** Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $[0, \pi/4]$.

40–45. Additional integrals *Evaluate the following integrals.*

40. $\int_0^{\sqrt{\pi/2}} x \sin^3 (x^2)\,dx$
41. $\int \dfrac{\sec^4 (\ln \theta)}{\theta}\,d\theta$
42. $\int_{\pi/6}^{\pi/2} \dfrac{dy}{\sin y}$
43. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\,d\theta$
44. $\int_{-\pi/4}^{\pi/4} \tan^3 x \sec^2 x\,dx$
45. $\int_0^{\pi} (1 - \cos 2x)^{3/2}\,dx$

46–49. Square roots *Evaluate the following integrals.*

46. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \cos 4x}\,dx$
47. $\int_0^{\pi/2} \sqrt{1 - \cos 2x}\,dx$
48. $\int_0^{\pi/8} \sqrt{1 - \cos 8x}\,dx$
49. $\int_0^{\pi/4} (1 + \cos 4x)^{3/2}\,dx$

50. **Sine football** Find the volume of the solid generated when the region bounded by the graph of $y = \sin x$ and the x-axis on the interval $[0, \pi]$ is revolved about the x-axis.
51. **Arc length** Find the length of the curve $y = \ln(\cos x)$ for $0 \le x \le \pi/4$.
52. **A sine reduction formula** Use integration by parts to obtain the following reduction formula for positive integers n:
$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx.$$
Then use an identity to obtain the reduction formula
$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$
Use this reduction formula to evaluate $\int \sin^6 x\,dx$.
53. **A tangent reduction formula** Prove that for positive integers $n \ne 1$,
$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$
Use the formula to evaluate $\int_0^{\pi/4} \tan^3 x\,dx$.
54. **A secant reduction formula** Prove that for positive integers $n \ne 1$,
$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$
(*Hint:* Integrate by parts with $u = \sec^{n-2} x$ and $dv = \sec^2 x\,dx$.)

Applications

55–59. Integrals of the form $\int \sin mx \cos nx\,dx$ *Use the following three identities to evaluate the given integrals.*

$$\sin mx \sin nx = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$
$$\sin mx \cos nx = \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x)]$$
$$\cos mx \cos nx = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$$

55. $\int \sin 3x \cos 7x\,dx$
56. $\int \sin 5x \sin 7x\,dx$
57. $\int \sin 3x \sin 2x\,dx$
58. $\int \cos x \cos 2x\,dx$
59. Prove the following **orthogonality relations** (which are used to generate *Fourier series*). Assume m and n are integers with $m \ne n$.
 a. $\int_0^{\pi} \sin mx \sin nx\,dx = 0$
 b. $\int_0^{\pi} \cos mx \cos nx\,dx = 0$
 c. $\int_0^{\pi} \sin mx \cos nx\,dx = 0$
60. **Mercator map projection** The Mercator map projection was proposed by the Flemish geographer Gerardus Mercator (1512–1594). The stretching of the Mercator map as a function of the latitude θ is given by the function
$$G(\theta) = \int_0^{\theta} \sec x\,dx.$$
Graph G for $0 \le \theta < \pi/2$. (See the Guided Projects for a derivation of this integral.)

Additional Exercises

61. **Exploring powers of sine and cosine**
 a. Graph the functions $f_1(x) = \sin^2 x$ and $f_2(x) = \sin^2 2x$ on the interval $[0, \pi]$. Find the area under these curves on $[0, \pi]$.
 b. Graph a few more of the functions $f_n(x) = \sin^2 nx$ on the interval $[0, \pi]$, where n is a positive integer. Find the area under these curves on $[0, \pi]$. Comment on your observations.
 c. Prove that $\int_0^{\pi} \sin^2 (nx)\,dx$ has the same value for all positive integers n.
 d. Does the conclusion of part (c) hold if sine is replaced by cosine?
 e. Repeat parts (a), (b), and (c) with $\sin^2 x$ replaced by $\sin^4 x$. Comment on your observations.
 f. Challenge problem: Show that for $m = 1, 2, 3, \ldots,$
$$\int_0^{\pi} \sin^{2m} x\,dx = \int_0^{\pi} \cos^{2m} x\,dx = \pi \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}.$$

QUICK CHECK ANSWERS

1. $\frac{1}{3}\cos^3 x - \cos x + C$ 2. Write $\int \sin^3 x \cos^3 x\,dx = \int \sin^2 x \cos^3 x \sin x\,dx = \int (1 - \cos^2 x)\cos^3 x \sin x\,dx$. Then, use the substitution $u = \cos x$. Or, begin by writing $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cos x\,dx$.